Indian Statistical Institute
B.Math.(Hons.) I Year
Second Semester Examination, 2004-2005
Algebra II

Time: 3 hrs | Date:16-05-05 | Total Marks : 50

Attempt questions 1 and 2 and any FOUR out of questions 3-7.

1. Let V be a finite dimensional vector space over a field F and let $T : V \to V$ be a linear transformation.

   a) If λ is an eigenvalue of T then prove that λ is also an eigenvalue of T^t (here T^t, is the transpose of T, considered as a linear transformation from V^* to V^*). [4 marks]

   b) Suppose T has the property that every subspace of V of dimension $= \dim V - 1$ is T-invariant. What can you say about T? [5 marks]

2. a) Let V be a finite dimensional vector space over $\mathbb{C}$ and let $T : V \to V$ be a linear transformation. Given any integer j such that $1 \leq j \leq \dim V$, prove that there exists a T-invariant subspace of V of dimension equal to j. Is this result true if we replace $\mathbb{C}$ by $\mathbb{R}$?

   [4 marks]

   b) Let $f(x)$ be a polynomial over a field F. Prove that there exists an $n \times n$ matrix with entries in F where characteristic polynomial is $f(x)$.

   [5 marks]

3. Let A be an $n \times n$ matrix over $\mathbb{C}$. Suppose every eigenvalue of A is real. Prove that A is similar to a matrix with real entries. [8 marks]

4. Let $A = (a_{ij})$ be an $n \times n$ matrix over $\mathbb{C}$ with $a_{ij} = 1\ \forall i, j$. What is the Jordan canonical form of A? [8 marks]

5. Determine all possible Jordan and rational canonical forms for a linear transformation whose characteristic polynomial is $(x-2)^3(x-3)^2$.

   [8 marks]

6. Let N be a $k \times k$ matrix with $N^k = 0$ but $N^{k-1} \neq 0$. Prove that N is similar to its transpose N^t. Use this to prove that any $n \times n$ complex matrix is similar to its transpose. [8 marks]

7. Let $R = \mathcal{C}([0,1],\ \mathbb{R})$ be the ring of continuous functions from [0,1] to $\mathbb{R}$.

   a) Show that there exists a bijective correspondence between all points of [0,1] and all maximal ideals of R via the mapping $x \in [0,1] \mapsto \mathcal{M}_x := \{f \in R / f(x) = 0\}$ [6 marks]

   b) Show that no maximal ideal of R is finitely generated. [2 marks]